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A normalized time-fractional Korteweg–de Vries equation

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ABSTRACT

A novel normalized time-fractional Korteweg–de Vries (KdV) equation is presented to investigate the effects of fractional time derivatives on nonlinear wave dynamics. The classical KdV model is extended by incorporating a fractional-order derivative, which captures memory and inherited properties in the evolution of soliton-like structures. Computational studies of the equation's nonlinear dynamics use a numerical scheme designed for the fractional temporal dimension. Simulations show that as the fractional parameter α decreases from 1 (the classical case) to smaller values, soliton dynamics change significantly. The soliton amplitude decreases, and its width increases. These changes are interpreted as dispersive or dissipative effects introduced by the fractional time component. At lower values of α, the soliton becomes broader and flatter, and its propagation is slowed. At intermediate values of α, multiple peaks and broader waveforms are observed, which implies more complex nonlinear interactions under fractional time evolution. The importance of fractional time derivatives in modifying the behavior of soliton solutions is highlighted, which demonstrates their potential in modeling physical systems where memory effects play a crucial role. The computational results provide insights into fractional partial differential equations and create new opportunities for future research in nonlinear wave propagation under fractional dynamics.

1. Introduction

The standard Korteweg–de Vries (KdV) equation has been applied to model nonlinear wave phenomena such as tsunamis, tidal waves, and internal ocean waves, where the wave height is relatively small compared to its wavelength [1–3]. These waves, known as solitons, can travel over long distances with minimal energy loss, maintain their shape, and move at constant speed [4]. The KdV equation has broad applications in atmospheric and oceanic wave theory [5], plasma physics [6], traffic flow [7,8], and biological systems. To account for effects such as anomalous dispersion or memory, the fractional KdV equation is often utilized [9,10].

Numerous studies have been conducted by modifying the standard KdV equation. For instance, Thamilmaran et al. [11] investigated rare phenomena in a damped KdV autonomous system using numerical simulations and experimental methods. Derakhshan and Aminataei [12] calculated the solution of the distributed-order time-fractional forced KdV equation using the Tau scheme. Similarly, Rehman et al. [13] modified the standard KdV equation using a space–time-fractional order and obtained soliton solutions through the Sardar-subequation method. Wang and Liu [14] developed an energy balance method in combination with the Sardar-subequation method to compute solutions of the nonlinear KdV equation for deep-water surface waves. These solutions included various wave forms, such as bright solitary waves, dark solitary waves, singular periodic waves, and perfect periodic waves. These solutions were represented using generalized hyperbolic functions, generalized trigonometric functions, and the cosine function. Ali et al. [15] applied the (G'/G)-expansion method to find traveling wave solutions of the space–time-fractional order KdV equation. Uddin et al. [16] investigated analytical soliton solutions of the space–time-fractional modified KdV equation using the generalized (G'/G)-expansion technique. Through the proposed method, diverse soliton solutions such as kink, periodic, and singular-kink types, were obtained. Liu and Zhang [17] calculated exact solutions of the nonlinear KdV model using an updated (G'/G)-expansion technique with space–time local fractional derivatives and found various forms of wave solutions such as dark soliton, explicit traveling wave, soliton, soliton-like, and periodic solutions. Moreover, Yousif et al. [9] used the conformable-Caputo fractional non-polynomial spline scheme to solve the time-fractional

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KdV equation and proved the stability of the method using the von Neumann method. Prakash et al. [18] proposed a modified He–Laplace method to solve nonlinear fractional equations in fields such as wave phenomena and nanotechnology. The method combines homotopy perturbation and Laplace transforms, which offers accurate, efficient, and simple solutions for various nonlinear equations. Yokuş [19] investigated exact wave solutions for the nonlinear KdV equation using the $(1/G')$-expansion technique. The results showed that, through an indexing technique, the computational solutions of the nonlinear time-fractional KdV equation closely approximated the analytic solution. Furthermore, von Neumann stability analysis was conducted for the applied method.

Similarly, Ameen et al. [20] extended the KdV equation using conformable fractional derivative and produced soliton waves with the software package Maple. Additionally, Sherriffe and Behera [21] found analytical traveling wave solutions in the form of solitons for a non-linear fifth-order time-fractional KdV equation using the sine–cosine method. Meanwhile, Li et al. [22] used an α-robust high-order computational algorithm to solve the time-fractional nonlinear KdV equation using the L1 formula based discretization technique on graded grids. The presented approach demonstrated the adaptability of fractional derivatives in capturing complex wave phenomena in applied physics and engineering. Aljohani [23] studied the fourth-order KdV–Klein/Gordon equation using Lie symmetries and conservation laws. Symmetry-based reduction techniques were applied to simplify the equation and derive conserved quantities, and numerical methods were used when analytical solutions were unattainable to emphasize the role of symmetries in complex wave phenomena. Khan et al. [24] investigated the nonlinear time-fractional KdV and modified KdV equations. They used an innovative approach called the Aboodh transform decomposition technique. Their findings revealed that the solitary wave solution for the time-fractional modified KdV model exhibits less stability against oscillations compared to the time-fractional KdV solution. To study the behavior of specific waves in nonlinear systems Hosseini et al. [25] used the Generalized Hirota Bilinear equation. Furthermore, in another study analyzing the long-wave behavior in shallow water, Hosseini et al. [26] used the three-dimensional KdV equation. The authors used the simplified Hirota method to retrieve multi-soliton waves. To analyze the behavior of water waves with long wavelengths, Umar et al. [27] investigated the two-dimensional generalized Kadomtsev–Petviashvili equation. By using the modified Hirota method, they derived single-, double-, and triple-soliton wave solutions after verifying the equation's integrability and satisfying the three-soliton condition.

Various disciplines are also incorporating fractal fractional derivatives into their research. For instance, Farman et al. [28] developed a fractional-order model based on an ABC-fractional-order dynamical system to analyze the effects of human-induced forest fires and explore sustainable forest resource management using a nonlinear mathematical framework. To track insulin and glucose levels in individuals experiencing stress, excitement, or trauma, Nisar et al. [29] introduced a novel fractional-order diabetes mellitus model. Additionally, a system of fractional differential equations was solved in [30] using a fractal fractional operator with a Mittag-Leffler-type kernel, which incorporates both fractal and fractional orders. Furthermore, Gokbulut et al. [31] proposed a nonlinear deterministic model to investigate the dynamics of Methicillin-resistant Staphylococcus aureus. Additionally, to solve time-fractional nonlinear porous medium equations Chew et al. [32] used the fractional Newton explicit group method. To study the transmission dynamics of the Nipah virus, Baleanu et al. [33] proposed a Caputo-type fractional model in which they studied the boundedness of the solution using the generalized fractional mean value theorem. However, the stability of the system was checked using the fractional Routh–Hurwitz criterion and LaSalle's invariance principle. To study the dynamics of the motion of an accelerated mass–spring system Defterli et al. [34] utilized a non-integer Euler–Lagrange model. Additionally, to analyze the tumor-immune surveillance mechanism, Baleanu et al. [35] presented a system of fractional differential equations and solved it using a numerical technique.

Nevertheless, when we investigate the effects of the order parameter α, it is challenging to make fair comparisons at the same time points, since the Caputo fractional derivative is not normalized in time. Therefore, the main purpose and novelty of this study are to introduce a normalized time-fractional derivative for the KdV equation. Lee et al. [36] presented a normalized time-fractional diffusion equation, which we extend to the KdV equation to explore the effects of α. We consider a new normalized time-fractional KdV equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = -6u(x,t)\frac{\partial u}{\partial x}(x,t) - \frac{\partial^3 u}{\partial x^3}(x,t), \tag{1}$$

where the normalized time-fractional derivative is given as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = \frac{1-\alpha}{t^{1-\alpha}}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}, \quad 0<\alpha<1. \tag{2}$$

Let

$$w_\alpha^t(s) = \frac{1-\alpha}{t^{1-\alpha}(t-s)^\alpha} \tag{3}$$

be a weight function. It is straightforward to verify that $W_\alpha(t) = \int_0^t w_\alpha^t(s)ds = 1$. Additional information regarding this weight function is available in [36]. Lazopoulos also derived Eq. (2), the L-fractional derivative, using a different approach, which has a significant geometrical interpretation, as discussed in [37,38].

The rest of this paper is organized as follows. Section 2 provides the numerical solution algorithm. Section 3 presents several computational tests, and Section 4 gives the conclusion. We provide the MATLAB code in the Appendix for interested readers.

2. Computational solution

We discretize the new normalized time-fractional KdV equation. Let $\Omega = [L_x, R_x]$. Let us define $u_i^n = u\left(L_x + (i-1)h, t_n\right)$, where $h = (R_x - L_x)/(N_x - 1)$ and $t_n = (n-1)\Delta t$, as shown in Fig. 1.

We discretize Eq. (2) as

$$\begin{aligned}\frac{\partial^\alpha u(x_i,t_{n+1})}{\partial t^\alpha} &= \frac{1-\alpha}{t_{n+1}^{1-\alpha}}\sum_{q=1}^n \int_{t_q}^{t_{q+1}} \frac{\partial u(x_i,s)}{\partial s}\frac{ds}{(t_{n+1}-s)^\alpha} \\ &\approx \sum_{q=1}^n \frac{1-\alpha}{t_{n+1}^{1-\alpha}}\int_{t_q}^{t_{q+1}}\frac{ds}{(t_{n+1}-s)^\alpha}\frac{u_i^{q+1}-u_i^q}{\Delta t} \\ &= \sum_{q=1}^n \frac{(n+1-q)^{1-\alpha}-(n-q)^{1-\alpha}}{n^{1-\alpha}}\frac{u_i^{q+1}-u_i^q}{\Delta t} = \sum_{q=1}^n w_q^n \frac{u_i^{q+1}-u_i^q}{\Delta t},\end{aligned} \tag{4}$$

where $w_q^n = \left[(n+1-q)^{1-\alpha} - (n-q)^{1-\alpha}\right]/n^{1-\alpha}$ and $\sum_{q=1}^n w_q^n = 1$. Using Eq. (4), we obtain

$$\sum_{q=1}^n w_q^n \frac{u_i^{q+1}-u_i^q}{\Delta t} = -6u_i^n\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2h} - \frac{-u_{i-2}^{n+1}+2u_{i-1}^{n+1}-2u_{i+1}^{n+1}+u_{i+2}^{n+1}}{2h^3}, \tag{5}$$

where we have used a finite difference method (FDM) [39]. Eq. (5) can be reformulated as follows: for $i = 3, \ldots, N_x - 2$,

$$\begin{aligned}-\frac{1}{2h^3}u_{i-2}^{n+1} + \left(\frac{1}{h^3} - \frac{3u_i^n}{h}\right)u_{i-1}^{n+1} + \frac{w_n^n}{\Delta t}u_i^{n+1} - \left(\frac{1}{h^3} - \frac{3u_i^n}{h}\right)u_{i+1}^{n+1} + \frac{1}{2h^3}u_{i+2}^{n+1} \\ = \frac{w_n^n}{\Delta t}u_i^n - \sum_{q=1}^{n-1} w_q^n \frac{u_i^{q+1}-u_i^q}{\Delta t}.\end{aligned} \tag{6}$$

We consider zero Dirichlet boundary conditions for simplicity of exposition:

$$u_1^n = 0,\ u_2^n = 0.5u_3^n,\ u_{N_x}^n = 0, \text{ and } u_{N_x-1}^n = 0.5u_{N_x-2}^n. \tag{7}$$

Eq. (6) can be written as

$$A\mathbf{u}^{n+1} = \mathbf{f},$$

where the parameters are given in Box I.

Eq. (6) can be solved using an efficient solver [40]. We may also apply a spectral method to solve the diffusion term [41].

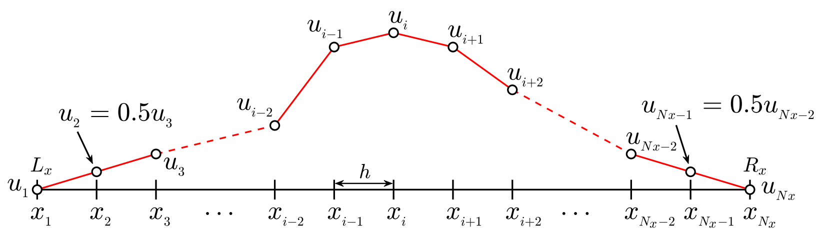

Fig. 1. Discrete computational domain.

$$A = \begin{pmatrix} \frac{w_n^n}{\Delta t} + \frac{1-3h^2u_3^n}{2h^3} & \frac{3h^2u_3^n-1}{h^3} & \frac{1}{2h^3} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1-3h^2u_4^n}{h^3} - \frac{1}{4h^3} & \frac{w_n^n}{\Delta t} & \frac{3h^2u_4^n-1}{h^3} & \frac{1}{2h^3} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{2h^3} & \frac{1-3h^2u_5^n}{h^3} & \frac{w_n^n}{\Delta t} & \frac{3h^2u_5^n-1}{h^3} & \frac{1}{2h^3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2h^3} & \frac{1-3h^2u_{N_x-2}^n}{h^3} & \frac{w_n^n}{\Delta t} + \frac{3h^2u_{N_x-2}^n-1}{2h^3} \end{pmatrix},$$

$$\mathbf{u}^{n+1} = \begin{pmatrix} u_3^{n+1} \\ u_4^{n+1} \\ \vdots \\ u_{N_x-2}^{n+1} \end{pmatrix}, \text{ and } \mathbf{f} = \begin{pmatrix} w_n^n \frac{u_3^n}{\Delta t} - F_3^n \\ w_n^n \frac{u_4^n}{\Delta t} - F_4^n \\ \vdots \\ w_n^n \frac{u_{N_x-2}^n}{\Delta t} - F_{N_x-2}^n \end{pmatrix} \text{ with } F_i^n = \sum_{p=1}^{n-1} w_p^n \frac{u_i^{p+1} - u_i^p}{\Delta t}.$$

Box I.

Table 1
The spatial convergence errors and ratios.

h	0.8		0.4		0.2
Error	5.0215e−3		1.0097e−3		2.4096e−4
Ratio		2.3141		2.0671	

Table 2
The temporal convergence errors and ratios.

Δt	1.6e−3		8.e-4		4.e-4
Error	5.6477e−5		2.6344e−5		1.1288e−5
Ratio		1.1002		1.2227	

3. Numerical experiments

We investigate the convergence of the numerical method. The initial condition is given on $\Omega = [0,\ 20]$ as

$$u(x,0) = \frac{1}{2}\text{sech}^2\left(\frac{x-8}{2}\right).$$

The parameters used for the spatial convergence test are $\Delta t = 5.\text{e-}5$, $\alpha = 0.1$, and $T = 1$. Table 1 presents the detailed values of spatial errors and the convergence rate. The error for h is defined as follows:

$$\text{Error} = \sqrt{\frac{1}{N_x}\sum_{i=1}^{N_x}\left(u_i - u^{\text{ref}}_{m(i-1)+1}\right)^2},$$

where u^{ref} is the reference solution and m is the ratio h/h^{ref}, where h^{ref} denotes the grid step size for the reference solution. To obtain the reference solution, we use $h^{\text{ref}} = 0.05$.

The temporal error and convergence rate are also investigated. The parameters used are the same as in the previous test. Here, $h = 0.2$ is fixed, and the reference solution is obtained using a time step size $\Delta t^{\text{ref}} = 1.\text{e-}4$. The error for Δt is defined as follows:

$$\text{Error} = \sqrt{\frac{1}{N_x}\sum_{i=1}^{N_x}\left(u_i - u_i^{\text{ref}}\right)^2},$$

Table 2 presents the convergence error values and convergence rate obtained as Δt decreases. As a result, we confirm that the numerical scheme is first-order in time and second-order in space.

We impose the following initial condition on $\Omega = [0,\ 20]$:

$$u(x,0) = \frac{1}{2}\text{sech}^2\left(\frac{x-8}{2}\right). \tag{8}$$

Then, the exact solution with $\alpha = 1$ is

$$u(x,t) = \frac{1}{2}\text{sech}^2\left(\frac{x-8-t}{2}\right). \tag{9}$$

We use $(N_x, N_t) = (101, 4000)$. Fig. 2 shows the numerical results for the normalized time-fractional KdV equation at time $t = 4$, with varying values of the fractional parameter α. Each subplot (a), (b), (c), and (d) shows the solution for a different α value with the same initial condition, which controls the degree of time-fractionality in the system. Fig. 2(a) represents the classical KdV equation, $\alpha = 1$. The blue curve shows the soliton structure after nonlinear interactions, while the red curve represents the initial condition. When $\alpha = 0.9$, the height of the soliton decreases, and the shape becomes slightly altered, which indicates the influence of fractional time on the system's evolution as shown in Fig. 2(b). At $\alpha = 0.5$, the waveform becomes broader and the peak height further decreases, which suggests that as α decreases, the intensity of the soliton diminishes as displayed in Fig. 2(c). For $\alpha = 0.1$, the soliton is much weaker, with a significantly reduced peak height and a broader base, and shows the pronounced effect of a highly fractional time parameter on the system, as observed in Fig. 2(d).

Next, we impose the following initial condition on $\Omega = [0,\ 30]$:

$$u(x,0) = 6\,\text{sech}^2(x-7). \tag{10}$$

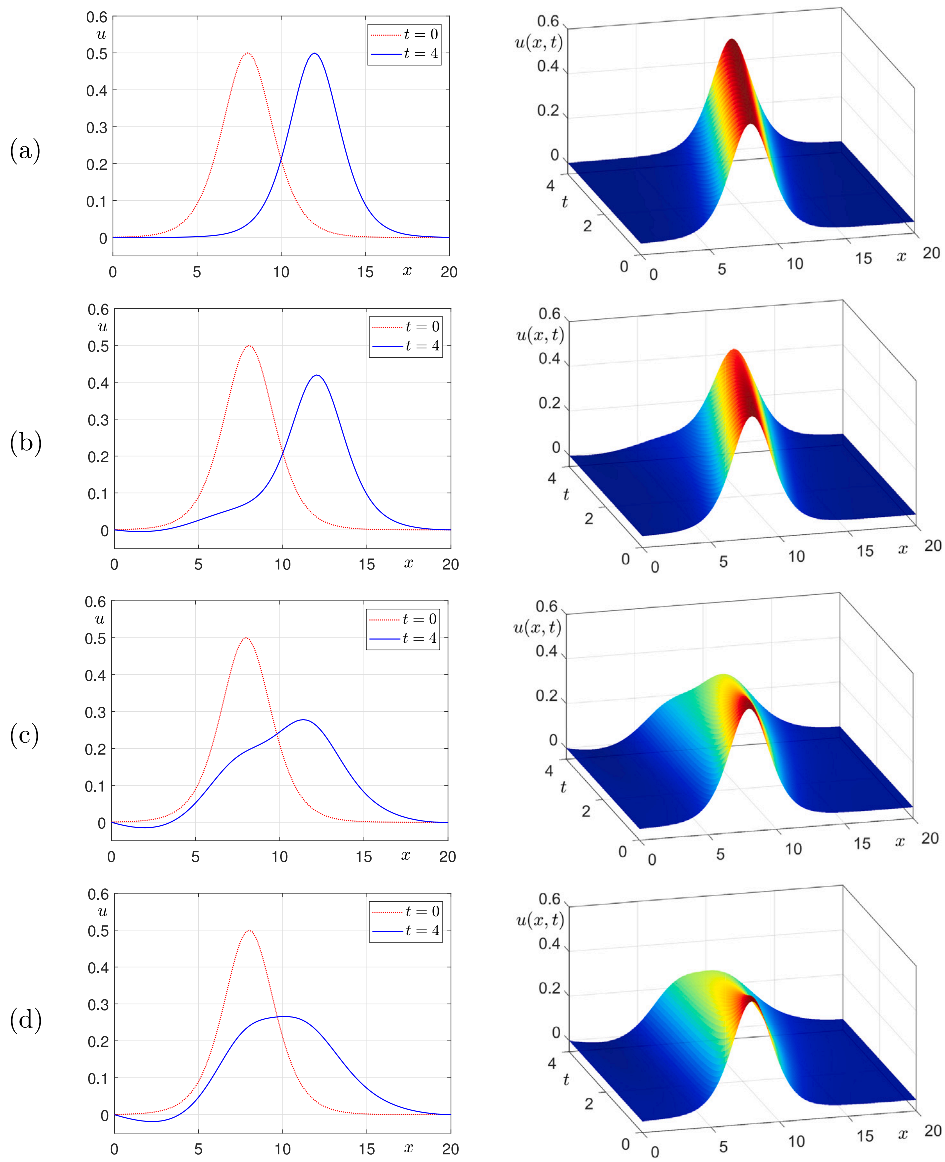

Fig. 2. (a), (b), (c), and (d) are the numerical solutions at time $t = 4$ with $\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.5$, and $\alpha = 0.1$, respectively. Here, $(N_x, N_t) = (101, 4000)$ is used.

Then, the exact solution with $\alpha = 1$ is as follows:

$$u(x,t) = 12\frac{3 + 4\cosh(2(x-7) - 8t) + \cosh(4(x-7) - 64t)}{[3\cosh(x - 7 - 28t) + \cosh(3(x-7) - 36t)]^2}. \quad (11)$$

Fig. 3 displays the numerical results for the normalized time-fractional KdV equation at time $t = 1$, with varying values of α. The subplots (a), (b), (c), and (d) correspond to different α values, ranging from $\alpha = 1$ to $\alpha = 0.1$, and show how the soliton evolves with the fractional parameter. The red dotted line shows the initial condition, and the blue line displays the numerical solution at $t = 1$. In Fig. 3(a), for $\alpha = 1$, which is the classical KdV case, the solution consists of a smooth soliton with a small peak near $x = 10$ and a sharp soliton with a high peak near $x = 22$. The soliton structure is well-preserved, which indicates strong nonlinearity and coherence of the wave over time. In Fig. 3(b), for $\alpha = 0.95$, the soliton begins to broaden, and multiple peaks appear in the regions near $x = 7,\ 12,\ 17$. This indicates a slight change in the soliton's behavior due to the fractional parameter. In Fig. 3(c), for $\alpha = 0.9$, the waveform further broadens, and shows multiple small peaks. The main soliton peak at $x = 16$ is lower compared to the previous cases, which reflects the increasing influence of the fractional time parameter and it tends to spread the energy over a wider region. In Fig. 3(d), when $\alpha = 0.1$, the solution is significantly dampened, with the main soliton peak drastically reduced in height and shifted closer to $x = 8$. The overall structure is much flatter, which indicates strong dissipative and memory effects from the small fractional parameter. As α decreases, the soliton disperses more and loses both height and coherence. This behavior indicates that the fractional time parameter induces dispersive or dissipative effects, particularly as α decreases, which causes the soliton propagation to slow down and its amplitude to decrease due to memory effects.

4. Conclusions

This study introduced a normalized time-fractional KdV equation to investigate the effects of fractional time derivatives on nonlinear wave dynamics. Through computational analysis, we demonstrated that as the fractional parameter α decreases, soliton amplitude diminishes, and the waveform broadens, which indicates the presence of dispersive or dissipative effects. For smaller values of α, solitons evolve into broader, flatter structures, which reflects the influence of fractional dynamics on wave propagation and coherence. These results highlight the significance of fractional time derivatives in altering soliton behavior and demonstrate their potential for modeling systems with inherent memory effects. By adopting a normalized time-fractional derivative, we

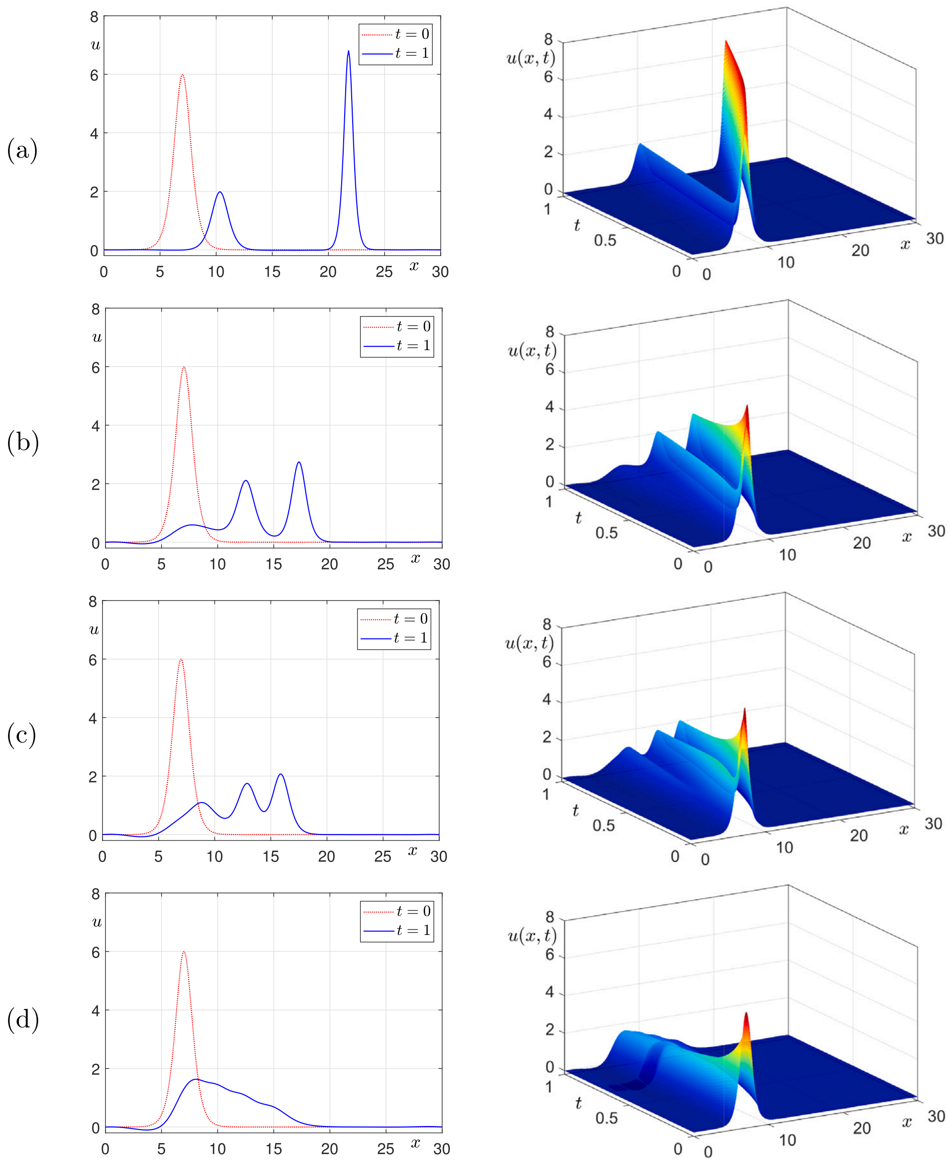

Fig. 3. (a), (b), (c), and (d) are the numerical solutions at time $t = 1$ with $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, and $\alpha = 0.1$, respectively. Here, $(N_x, N_t) = (301, 10000)$ is used.

enabled a fair comparison of the effects of different fractional orders. Future research could explore various directions, including examining soliton interactions for different fractional orders, incorporating alternative fractional derivatives such as Caputo or Riesz derivatives, and extending the model to triangular grids [42] and higher-dimensional spaces, including two- and three-dimensional domains, to analyze more complex wave phenomena. Furthermore, applying this model to real-world systems such as water waves or plasmas may provide practical insights into how fractional dynamics influence physical systems with memory effects.

CRediT authorship contribution statement

Hyun Geun Lee: Writing – review & editing, Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Soobin Kwak:** Writing – review & editing, Writing – original draft, Software, Resources, Methodology, Investigation. **Jyoti:** Writing – review & editing, Writing – original draft, Validation, Investigation, Formal analysis. **Yunjae Nam:** Writing – review & editing, Writing – original draft, Visualization, Validation, Investigation. **Junseok Kim:** Writing – review & editing, Writing – original draft, Validation, Supervision, Project administration, Methodology, Funding acquisition, Formal analysis, Conceptualization.

Use of AI tools declaration

The authors have not used Artificial Intelligence (AI) tools in the creations of this article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Listing 1: KdV code

```
clear; clf; Nx=101; x=linspace(0,20,Nx); h=x(2)-x(1);
T=4; dt=0.025*h^2; Nt=round(T/dt); dt=T/Nt;
for i=1:Nx
    a(i)=-1.0/(2.0*h^3); e(i)=1.0/(2.0*h^3);
end
u(:,1)=0.5*(sech(0.5*(x-8))).^2;
flag=1;  % flag=1, alpha=1,   flag=2, 0<alpha<1
alphaset=[1 0.9 0.5 0.1];
for k=1:4
    alp=alphaset(k);
if alp==1
    flag=1;
else
    flag=2;
end
for m=1:Nt
    w(m)=1;
    if flag==2
        for q = 1:m
            w(q) = ((m+1-q)^(1-alp)-(m-q)^(1-alp))/m^(1-alp);
        end
        F = zeros(Nx, 1);
        if m > 1
            for q = 1:m-1
                F = F+w(q)*(u(:,q+1)-u(:,q))/dt;
            end
        end
    end
    for i=1:Nx
        b(i)=1.0/h^3-3*u(i,m)/h; c(i)=w(m)/dt;
        d(i)=-1.0/h^3+3*u(i,m)/h;
    end
    c(3)=c(3)+0.5*b(3); b(4)=b(4)+0.5*a(4);
    c(Nx-2)=c(Nx-2)+0.5*d(Nx-2); d(Nx-3)=d(Nx-3)+0.5*e(Nx-3);
    f = w(m)/dt*u(:,m);
    if flag==2
        f=f-F;
    end
    u(3:Nx-2,m+1)=thomas_penta(a(3:Nx-2), b(3:Nx-2), ...
            c(3:Nx-2), d(3:Nx-2), e(3:Nx-2), f(3:Nx-2));
    u(2,m+1)=0.5*u(3,m+1); u(Nx-1,m+1)=0.5*u(Nx-2,m+1);
    if mod(m,500)==0
    clf; plot(x, u(:,1),'r-'); hold on; plot(x, u(:,m+1),'b-');
    grid on; axis([x(1) x(end) -0.1 0.6]); pause(0.01)
    end
end
    uu{k}=u(:,Nt+1);
end
for k=1:4
figure(k); clf; plot(x,u(:,1),'r:','LineWidth',1.5); hold on
plot(x,uu{k},'b-','LineWidth',1.5)
axis([x(1) x(end) -0.05 0.6]); box on; grid on
xlabel('x'); ylabel('u')
end
function x = thomas_penta(a, b, c, d, e, f)
% solve Ax=bb
% [ c d e
%   b c d e
%   a b c d e         [x]=[f]
%   0 a b c d e ...
%   ...               ]
Nx = length(c); x=zeros(1,Nx);
for i=2:Nx-1
    xmult=b(i)/c(i-1); c(i)=c(i)-xmult*d(i-1);
    d(i)=d(i)-xmult*e(i-1); f(i)=f(i)-xmult*f(i-1);
    xmult=a(i+1)/c(i-1); b(i+1)=b(i+1)-xmult*d(i-1);
    c(i+1)=c(i+1)-xmult*e(i-1); f(i+1)=f(i+1)-xmult*f(i-1);
end
xmult=b(Nx)/c(Nx-1); c(Nx)=c(Nx)-xmult*d(Nx-1);
x(Nx)=(f(Nx)-xmult*f(Nx-1))/c(Nx);
x(Nx-1)=(f(Nx-1)-d(Nx-1)*x(Nx))/c(Nx-1);
for i=Nx-2:-1:1
    x(i)=(f(i)-e(i)*x(i+2)-d(i)*x(i+1))/c(i);
end
end
```

Data availability

No data was used for the research described in the article.

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