

A DISCRETE SINE TRANSFORM METHOD FOR THE NORMALIZED TIME-FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. In this study, we propose a discrete sine transform (DST) method for the two-dimensional normalized time-fractional diffusion equation with homogeneous Dirichlet boundary conditions. The normalized time-fractional derivative is approximated on a uniform time grid. For the spatial discretization, we use an orthonormal DST. This choice is natural because the sine basis satisfies the homogeneous Dirichlet boundary condition. In the sine space, the Laplace operator becomes diagonal. Therefore each spectral mode can be computed independently, and a large two-dimensional linear system is not required. We prove that the proposed method is unconditionally stable in the discrete ℓ^2 norm. The method has a temporal convergence order of $2-\alpha$ for sufficiently smooth solutions and spectral accuracy in space. Numerical experiments support the theoretical results. They also show that the fractional order affects the decay speed of the solution. In particular, smaller values of α lead to slower decay in the tested cases.

1. INTRODUCTION

Time-fractional differential equations have been widely studied because they can describe memory effects. In these models, the present state depends on the past history of the solution. The fractional order controls the strength of this effect. Time-fractional diffusion equations have therefore been used in many areas of science and engineering.

Recently, a normalized time-fractional derivative has been considered to give a clearer meaning to the fractional-order effect [1, 2, 3]. Its main feature is simple. The total weight of the memory kernel is always one. Based on this idea, Lee et al. [3] studied a normalized time-fractional diffusion equation and used a finite difference method for the numerical solution. The normalized derivative has also been applied to other equations, including the Fisher equation [4] and the Black–Scholes equation [5].

More recently, Zhang et al. [6] studied normalized time-fractional diffusion equations in two and three dimensions. They used a Fourier spectral method in space and a finite difference

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method in time. Their method showed a temporal convergence rate of $\mathcal{O}(\Delta t^{2-\alpha})$ and high spatial accuracy. Periodic boundary conditions were used in their numerical simulations. This method gives an efficient way to solve multidimensional normalized time-fractional diffusion equations.

Spectral methods are useful when the solution is smooth. They can give high spatial accuracy with a relatively small number of grid points. The choice of boundary condition is important because it can affect both the numerical behavior and the physical interpretation of the solution [7]. The choice of the transform is closely related to the boundary condition. For periodic boundary conditions, Fourier basis functions are commonly used, and the fast Fourier transform (FFT) [8] provides an efficient computation. For homogeneous Neumann boundary conditions, cosine functions are natural basis functions, and the discrete cosine transform (DCT) can be used [9]. For homogeneous Dirichlet boundary conditions, sine functions are suitable because they vanish at the boundary. In this case, the discrete sine transform (DST) is a natural choice. The exact form of each transform can depend on the grid and the transform type.

In this study, we consider the two-dimensional normalized time-fractional diffusion equation with homogeneous Dirichlet boundary conditions. Unlike the Fourier spectral formulation in [6], for which periodic boundary conditions were adopted in the numerical experiments, we use an orthonormal DST that is naturally adapted to homogeneous Dirichlet boundary conditions. The sine basis satisfies the boundary condition directly and diagonalizes the Laplace operator. Therefore, each spectral mode can be updated independently without the direct solution of a large two-dimensional linear system.

The main contribution of the present study is a Dirichlet-adapted spectral formulation for the normalized time-fractional diffusion equation, together with its discrete ℓ^2 stability and convergence analyses. Numerical experiments verify the temporal and spatial accuracy of the proposed method. They also show the effect of the fractional order.

The remainder of this paper is organized as follows. Section 2 presents the numerical method. Section 3 gives the stability and convergence analyses. Numerical experiments are presented in Section 4. Finally, Section 5 gives the conclusions.

2. NUMERICAL METHOD

In this section, we present a numerical method for the two-dimensional normalized time-fractional diffusion equation. We consider a computational domain $\Omega = (L_x, R_x) \times (L_y, R_y)$ and a final time $T > 0$. The governing equation is given by

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = \Delta u & (x, y) \in \Omega, \quad 0 < t \leq T, \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \\ u(x, y, t) = 0, & (x, y) \in \partial\Omega, \quad 0 < t \leq T, \end{cases} \quad (2.1)$$

where $0 < \alpha < 1$ is the fractional order, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator.

The time-fractional derivative in Eq. (2.1) is the normalized time-fractional derivative considered in [1, 2, 5]. It is defined by

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{1 - \alpha}{t^{1-\alpha}} \int_0^t \frac{\partial u(x, y, s)}{\partial s} \frac{ds}{(t - s)^\alpha}. \quad (2.2)$$

The factor $(1 - \alpha)/t^{1-\alpha}$ normalizes the weight function. It satisfies

$$\frac{1 - \alpha}{t^{1-\alpha}} \int_0^t \frac{ds}{(t - s)^\alpha} = 1.$$

Thus, the total weight is one for any $0 < \alpha < 1$ and $t > 0$.

We first discretize the normalized time-fractional derivative. Let the time interval $[0, T]$ be divided into N_t uniform intervals. We define $t_n = n\Delta t$ where $n = 0, 1, \dots, N_t$ and $\Delta t = \frac{T}{N_t}$. At $t = t_{n+1}$, Eq. (2.2) can be written as

$$\frac{\partial^\alpha u(x, y, t_{n+1})}{\partial t^\alpha} = \frac{1 - \alpha}{t_{n+1}^{1-\alpha}} \sum_{k=0}^n \int_{t_k}^{t_{k+1}} \frac{\partial u(x, y, s)}{\partial s} \frac{ds}{(t_{n+1} - s)^\alpha}.$$

On each time interval $[t_k, t_{k+1}]$, we use

$$\frac{\partial u(x, y, s)}{\partial s} \approx \frac{u^{k+1}(x, y) - u^k(x, y)}{\Delta t}.$$

This gives

$$\frac{\partial^\alpha u(x, y, t_{n+1})}{\partial t^\alpha} \approx \sum_{k=0}^n w_k^{n+1} \frac{u^{k+1}(x, y) - u^k(x, y)}{\Delta t},$$

where

$$w_k^{n+1} = \frac{(n + 1 - k)^{1-\alpha} - (n - k)^{1-\alpha}}{(n + 1)^{1-\alpha}}, \quad k = 0, 1, \dots, n. \quad (2.3)$$

The normalized weights satisfy

$$\sum_{k=0}^n w_k^{n+1} = 1. \quad (2.4)$$

For the spatial approximation, we introduce the uniform grid points $x_i = L_x + ih_x$, where $h_x = (R_x - L_x)/N_x$ and $i = 0, 1, \dots, N_x$, and $y_j = L_y + jh_y$, where $h_y = (R_y - L_y)/N_y$ and $j = 0, 1, \dots, N_y$. Let u_{ij}^n denote the numerical approximation to $u(x_i, y_j, t_n)$. Since u satisfies the homogeneous Dirichlet boundary condition, we use sine functions as the spatial basis.

We use the discrete sine transform (DST). The spectral coefficients are defined by

$$\hat{u}_{pq}^n = \frac{2}{\sqrt{N_x N_y}} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} u_{ij}^n \sin\left(\frac{ip\pi}{N_x}\right) \sin\left(\frac jq\pi}{N_y}\right),$$

where $p = 1, \dots, N_x - 1$ and $q = 1, \dots, N_y - 1$. The inverse DST is

$$u_{ij}^n = \frac{2}{\sqrt{N_x N_y}} \sum_{p=1}^{N_x-1} \sum_{q=1}^{N_y-1} \hat{u}_{pq}^n \sin\left(\frac{ip\pi}{N_x}\right) \sin\left(\frac jq\pi}{N_y}\right).$$

For each sine mode, the Laplace operator satisfies

$$\Delta \left[\sin\left(\frac{p\pi(x-L_x)}{R_x-L_x}\right) \sin\left(\frac{q\pi(y-L_y)}{R_y-L_y}\right) \right] = -\lambda_{pq} \sin\left(\frac{p\pi(x-L_x)}{R_x-L_x}\right) \sin\left(\frac{q\pi(y-L_y)}{R_y-L_y}\right),$$

where $\lambda_{pq} = (p\pi/(R_x - L_x))^2 + (q\pi/(R_y - L_y))^2$.

We now combine the normalized time discretization with the sine spectral method. For each spectral mode (p, q) , we obtain

$$\sum_{k=0}^n w_k^{n+1} \frac{\hat{u}_{pq}^{k+1} - \hat{u}_{pq}^k}{\Delta t} = -\lambda_{pq} \hat{u}_{pq}^{n+1}. \quad (2.5)$$

Separating the last term in the time summation gives

$$\left(\frac{w_n^{n+1}}{\Delta t} + \lambda_{pq} \right) \hat{u}_{pq}^{n+1} = \frac{w_n^{n+1}}{\Delta t} \hat{u}_{pq}^n - \sum_{k=0}^{n-1} w_k^{n+1} \frac{\hat{u}_{pq}^{k+1} - \hat{u}_{pq}^k}{\Delta t}.$$

Therefore,

$$\hat{u}_{pq}^{n+1} = \frac{\frac{w_n^{n+1}}{\Delta t} \hat{u}_{pq}^n - \sum_{k=0}^{n-1} w_k^{n+1} \frac{\hat{u}_{pq}^{k+1} - \hat{u}_{pq}^k}{\Delta t}}{\frac{w_n^{n+1}}{\Delta t} + \lambda_{pq}}. \quad (2.6)$$

For $n = 0$, the summation on the right-hand side is zero. Equation (2.6) determines each spectral coefficient independently. Thus, a large two-dimensional linear system does not need to be solved. The numerical solution u_{ij}^{n+1} is obtained from the inverse DST.

Remark 2.1. Let $M = (N_x - 1)(N_y - 1)$ be the number of interior grid points. A direct evaluation of the fractional history term requires $\mathcal{O}(MN_t^2)$ operations over all time steps. The storage cost is $\mathcal{O}(MN_t)$ if all previous spectral coefficients are stored. These costs are common to direct LI-type methods.

The main advantage of the DST method appears in the spatial calculation. The Laplace operator is diagonal in the sine space. Therefore, each spectral mode is updated by a scalar equation. The method does not require a large two-dimensional linear system at each time step. A conventional implicit finite difference method generally requires such a spatial solve. Fast history algorithms could further reduce the temporal cost, but this topic is beyond the scope of the present study.

3. STABILITY AND CONVERGENCE ANALYSIS

3.1. Stability analysis. In this subsection, we study the stability of the proposed numerical method. From Eq. (2.3), the normalized weights are positive and satisfy

$$0 < w_0^{n+1} < w_1^{n+1} < \dots < w_n^{n+1}.$$

Indeed, the function $s^{1-\alpha}$ is increasing and concave for $0 < \alpha < 1$. Therefore, the difference $s^{1-\alpha} - (s-1)^{1-\alpha}$ decreases as s increases. Since $n+1-k$ decreases as k increases, the weights increase with k . In addition, Eq. (2.4) gives

$$\sum_{k=0}^n w_k^{n+1} = 1.$$

We consider two numerical solutions u_{ij}^n and $\tilde{u}_{ij}^n$ obtained from Eq. (2.5). They satisfy the same homogeneous boundary condition, but their initial values may be different. We define the perturbation by

$$e_{ij}^n = u_{ij}^n - \tilde{u}_{ij}^n.$$

Let $\hat{e}_{pq}^n$ denote its DST coefficient. By subtracting the two numerical schemes, we obtain

$$\sum_{k=0}^n w_k^{n+1} \frac{\hat{e}_{pq}^{k+1} - \hat{e}_{pq}^k}{\Delta t} = -\lambda_{pq} \hat{e}_{pq}^{n+1}. \quad (3.1)$$

Multiplying Eq. (3.1) by Δt and rearranging the terms gives

$$(w_n^{n+1} + \Delta t \lambda_{pq}) \hat{e}_{pq}^{n+1} = w_0^{n+1} \hat{e}_{pq}^0 + \sum_{k=1}^n (w_k^{n+1} - w_{k-1}^{n+1}) \hat{e}_{pq}^k. \quad (3.2)$$

All coefficients on the right-hand side are nonnegative. Moreover,

$$w_0^{n+1} + \sum_{k=1}^n (w_k^{n+1} - w_{k-1}^{n+1}) = w_n^{n+1}.$$

Theorem 3.1. *The proposed numerical method is unconditionally stable in the discrete ℓ^2 norm. More precisely,*

$$\|e^n\|_h \leq \|e^0\|_h, \quad n = 1, 2, \dots, N_t,$$

where

$$\|e^n\|_h = \left(h_x h_y \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} |e_{ij}^n|^2 \right)^{1/2}.$$

No restriction on the time step or the spatial resolution is required.

Proof. We first consider each spectral mode. For $n = 0$, Eq. (3.1) gives

$$(w_0^1 + \Delta t \lambda_{pq}) \hat{e}_{pq}^1 = w_0^1 \hat{e}_{pq}^0.$$

Since $w_0^1 = 1$ and $\lambda_{pq} > 0$, we have

$$|\hat{e}_{pq}^1| \leq |\hat{e}_{pq}^0|.$$

Assume that

$$|\hat{e}_{pq}^k| \leq |\hat{e}_{pq}^0|, \quad k = 1, 2, \dots, n.$$

Using Eq. (3.2), we obtain

$$\begin{aligned} |\hat{e}_{pq}^{n+1}| &\leq \frac{w_0^{n+1} |\hat{e}_{pq}^0| + \sum_{k=1}^n (w_k^{n+1} - w_{k-1}^{n+1}) |\hat{e}_{pq}^k|}{w_n^{n+1} + \Delta t \lambda_{pq}} \\ &\leq \frac{w_n^{n+1}}{w_n^{n+1} + \Delta t \lambda_{pq}} |\hat{e}_{pq}^0| \\ &\leq |\hat{e}_{pq}^0|. \end{aligned} \tag{3.3}$$

Thus, the inequality holds for all n by induction.

Since the DST is orthonormal, the Parseval identity gives

$$\sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} |e_{ij}^n|^2 = \sum_{p=1}^{N_x-1} \sum_{q=1}^{N_y-1} |\hat{e}_{pq}^n|^2.$$

Using Eq. (3.3), we obtain

$$\|e^n\|_h^2 \leq \|e^0\|_h^2.$$

Hence,

$$\|e^n\|_h \leq \|e^0\|_h.$$

This completes the proof. $\square$

3.2. Convergence analysis. In this subsection, we study the convergence of the proposed method. We first consider the error from the time discretization. For convenience, let $\mathcal{D}_t^\alpha$ denote the normalized time-fractional derivative defined in Eq. (2.2).

The normalized time-fractional derivative is related to the Caputo derivative. For $0 < \alpha < 1$, we have

$$\mathcal{D}_t^\alpha u(t) = \Gamma(2 - \alpha) t^{\alpha-1} {}^C D_t^\alpha u(t),$$

where

$${}^C D_t^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\partial u(s)}{\partial s} \frac{ds}{(t - s)^\alpha}.$$

The spatial variables are omitted here for simplicity.

Let $a_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ for $j = 0, 1, \dots$. Then, the normalized weights can be written as

$$w_k^{n+1} = \frac{a_{n-k}}{(n+1)^{1-\alpha}}. \quad (3.4)$$

The standard L1 approximation of the Caputo derivative at $t = t_{n+1}$ is

$${}^C\delta_t^\alpha u^{n+1} = \frac{1}{\Gamma(2-\alpha)(\Delta t)^\alpha} \sum_{k=0}^n a_{n-k} (u^{k+1} - u^k).$$

From Eq. (3.4), the normalized time discretization can be written as

$$\sum_{k=0}^n w_k^{n+1} \frac{u^{k+1} - u^k}{\Delta t} = \Gamma(2-\alpha) t_{n+1}^{\alpha-1} {}^C\delta_t^\alpha u^{n+1}.$$

Thus, the proposed temporal discretization is a normalized form of the L1 approximation.

Suppose that u is sufficiently smooth in time and that $\partial^2 u / \partial t^2$ is bounded. The L1 approximation satisfies

$$\| {}^C D_t^\alpha u(t_{n+1}) - {}^C\delta_t^\alpha u^{n+1} \| \leq C(\Delta t)^{2-\alpha},$$

where C is independent of Δt . Thus, the L1 formula has the temporal consistency order $2-\alpha$ for a sufficiently smooth solution.

We now consider the global temporal error. Let $P_{N_x, N_y} u$ denote the sine spectral projection of u onto the finite-dimensional sine space used in the numerical method. The projection and the DST use the same sine basis. The projection coefficients are defined by continuous inner products, whereas the DST coefficients are computed from the grid values. The projection is used here as an auxiliary function for the error analysis.

Let $\hat{U}_{pq}(t)$ denote the (p, q) -th coefficient of $P_{N_x, N_y} u(t)$. For each sine mode, the projected equation satisfies

$$\mathcal{D}_t^\alpha \hat{U}_{pq}(t) = -\lambda_{pq} \hat{U}_{pq}(t).$$

From the relation between the normalized and Caputo derivatives, this equation becomes

$${}^C D_t^\alpha \hat{U}_{pq}(t) + \mu_{pq}(t) \hat{U}_{pq}(t) = 0,$$

where

$$\mu_{pq}(t) = \frac{\lambda_{pq} t^{1-\alpha}}{\Gamma(2-\alpha)} \geq 0.$$

The fully discrete scheme for the same mode is

$${}^C\delta_t^\alpha \hat{u}_{pq}^{n+1} + \mu_{pq}(t_{n+1}) \hat{u}_{pq}^{n+1} = 0.$$

We define the temporal error for each mode by

$$\eta_{pq}^n = \hat{U}_{pq}(t_n) - \hat{u}_{pq}^n.$$

Then, the error equation is

$${}^C\delta_t^\alpha \eta_{pq}^{n+1} + \mu_{pq}(t_{n+1})\eta_{pq}^{n+1} = \rho_{pq}^{n+1},$$

where ρ_{pq}^{n+1} is the local truncation error of the L1 approximation. For a sufficiently smooth solution,

$$|\rho_{pq}^{n+1}| \leq C(\Delta t)^{2-\alpha}.$$

Since $\mu_{pq}(t) \geq 0$, the same positivity argument used in the standard L1 stability analysis [10] gives

$$\|P_{N_x, N_y} u(t_n) - u^n\|_h \leq C [\|P_{N_x, N_y} u(0) - u^0\|_h + (\Delta t)^{2-\alpha}]. \quad (3.5)$$

We next consider the spatial approximation. A sine spectral approximation requires smoothness that is compatible with the homogeneous Dirichlet boundary condition. We assume that the odd extension of $u(\cdot, \cdot, t)$ across the boundaries is m times continuously differentiable for some $m \geq 2$. This assumption includes the required compatibility of the solution and its derivatives at the boundary.

Under this assumption, the sine spectral projection satisfies

$$\|u(\cdot, \cdot, t) - P_{N_x, N_y} u(\cdot, \cdot, t)\|_h \leq C (N_x^{-m} + N_y^{-m}). \quad (3.6)$$

The constant C is independent of N_x and N_y .

Let

$$\varepsilon_{ij}^n = u(x_i, y_j, t_n) - u_{ij}^n$$

denote the numerical error at (x_i, y_j, t_n) .

Theorem 3.2. *Suppose that the exact solution u is sufficiently smooth in time and space. Assume that*

$$u \in C^2([0, T]; L^2(\Omega)).$$

We also assume that the odd extension of $u(\cdot, \cdot, t)$ across the boundaries is m times continuously differentiable for some $m \geq 2$. Then, for a fixed $t_n > 0$, the numerical solution satisfies

$$\|\varepsilon^n\|_h \leq C [(\Delta t)^{2-\alpha} + N_x^{-m} + N_y^{-m}],$$

where C is independent of Δt , N_x , and N_y .

Proof. We divide the error into a spatial projection error and a temporal discretization error. At the grid points, we write

$$\varepsilon_{ij}^n = [u(x_i, y_j, t_n) - (P_{N_x, N_y} u)(x_i, y_j, t_n)] + [(P_{N_x, N_y} u)(x_i, y_j, t_n) - u_{ij}^n].$$

The first term is bounded by the sine spectral approximation estimate in Eq. (3.6). By Eq. (3.6) at $t = 0$,

$$\|P_{N_x, N_y} u(0) - u^0\|_h \leq C (N_x^{-m} + N_y^{-m}).$$

Therefore, Eq. (3.5) gives

$$\|P_{N_x, N_y} u(t_n) - u^n\|_h \leq C [(\Delta t)^{2-\alpha} + N_x^{-m} + N_y^{-m}].$$

The triangle inequality then gives

$$\|\varepsilon^n\|_h \leq C [(\Delta t)^{2-\alpha} + N_x^{-m} + N_y^{-m}].$$

This completes the proof. $\square$

Remark 3.3. *The temporal convergence order depends on the fractional order α . The expected order is $2 - \alpha$. Thus, the order approaches two as α approaches zero and approaches one as α approaches one. For example, the expected orders are 1.8, 1.5, and 1.2 for $\alpha = 0.2, 0.5$, and 0.8 , respectively. These rates are confirmed by the numerical results in Section 4.1.*

Therefore, for a sufficiently smooth solution, the proposed method has an order of $2 - \alpha$ in time and spectral accuracy in space.

4. NUMERICAL EXPERIMENTS

4.1. Convergence test. We first test the temporal convergence of the proposed method. The computational domain is $\Omega = (-1, 1) \times (-1, 1)$ and the final time is $T = 0.05$. We consider three fractional orders, $\alpha = 0.2, 0.5$, and 0.8 and fix the spatial resolution as $N_x = N_y = 64$. The initial condition is given by

$$u(x, y, 0) = \sin\left(\frac{\pi(x - L_x)}{R_x - L_x}\right) \sin\left(\frac{\pi(y - L_y)}{R_y - L_y}\right) \exp\left[\cos\left(\frac{2\pi(x - L_x)}{R_x - L_x}\right) + \cos\left(\frac{2\pi(y - L_y)}{R_y - L_y}\right)\right].$$

This initial condition is smooth and satisfies the homogeneous Dirichlet boundary condition.

We use the time step sizes $\Delta t = 0.01, 0.005, 0.0025, 0.00125, 0.000625$. Since an exact solution is not available, a reference solution is computed with $\Delta t = 0.01/2^8$. The numerical errors are measured at $T = 0.05$. We use the discrete ℓ^2 and ℓ^∞ errors defined by

$$E_2 = \left[\frac{1}{(N_x - 1)(N_y - 1)} \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} |u_{ij} - u_{ij}^{\text{ref}}|^2 \right]^{1/2}, \quad \text{and} \quad E_\infty = \max_{i,j} |u_{ij} - u_{ij}^{\text{ref}}|.$$

The convergence rate is calculated by

$$\text{Rate} = \log_2 \left(\frac{E(\Delta t)}{E(\Delta t/2)} \right).$$

Table 1 shows the temporal convergence results for $\alpha = 0.2, 0.5$, and 0.8 . The theoretical temporal convergence order is $2 - \alpha$, which corresponds to 1.8, 1.5, and 1.2, respectively. For $\alpha = 0.2$, the observed rates approach 1.8 as the time step decreases. For $\alpha = 0.5$, the observed rates are approximately 1.5. For $\alpha = 0.8$, the rates are close to 1.2. Both the ℓ^2 and ℓ^∞ errors exhibit the expected convergence behavior for all three fractional orders. These results confirm the theoretical temporal convergence order $\mathcal{O}(\Delta t^{2-\alpha})$.

We next test the spatial convergence of the proposed method for the same three fractional orders, $\alpha = 0.2, 0.5$, and 0.8 . We use the same computational domain, initial condition, and

Table 1: Temporal convergence errors and rates at $T = 0.05$ for $\alpha = 0.2, 0.5$, and 0.8 . The spatial resolution is fixed at $N_x = N_y = 64$, and the reference solution for each fractional order is computed with $\Delta t = 0.01/2^8$.

α	Δt	ℓ^2 error	Rate	ℓ^∞ error	Rate
0.2	0.01	2.9040×10^{-4}	—	6.8324×10^{-4}	—
	0.005	8.5707×10^{-5}	1.76	2.0033×10^{-4}	1.77
	0.0025	2.5236×10^{-5}	1.76	5.8627×10^{-5}	1.77
	0.00125	7.4003×10^{-6}	1.77	1.7096×10^{-5}	1.78
	0.000625	2.1558×10^{-6}	1.78	4.9558×10^{-6}	1.79
0.5	0.01	1.4954×10^{-3}	—	3.4607×10^{-3}	—
	0.005	5.3156×10^{-4}	1.49	1.2197×10^{-3}	1.50
	0.0025	1.8815×10^{-4}	1.50	4.2969×10^{-4}	1.51
	0.00125	6.6379×10^{-5}	1.50	1.5115×10^{-4}	1.51
	0.000625	2.3253×10^{-5}	1.51	5.2843×10^{-5}	1.52
0.8	0.01	5.0640×10^{-3}	—	1.1888×10^{-2}	—
	0.005	2.2427×10^{-3}	1.18	5.2128×10^{-3}	1.19
	0.0025	9.7904×10^{-4}	1.20	2.2617×10^{-3}	1.20
	0.00125	4.2332×10^{-4}	1.21	9.7464×10^{-4}	1.21
	0.000625	1.8058×10^{-4}	1.23	4.1504×10^{-4}	1.23

final time as in the previous test. The time step is fixed as $\Delta t = 10^{-4}$, so that the temporal error is sufficiently small. We take $N_x = N_y = N = 8, 16, 32, 64, 128$. Since the exact solution is not available, the numerical solution with $N = 1024$ is used as the reference solution. The reference and test solutions are computed with the same time step, $\Delta t = 10^{-4}$, and the errors are measured at $T = 0.05$.

Figure 1 shows the ℓ^2 and ℓ^∞ errors on a semilogarithmic scale for $\alpha = 0.2, 0.5$, and 0.8 . For all three fractional orders, the errors decrease rapidly as the spatial resolution increases. The errors eventually reach approximately machine precision, after which no further systematic decrease is observed because round-off errors become dominant. The similar behavior for the three values of α confirms that the high spatial accuracy of the sine spectral method is maintained over the tested range of fractional orders. Because the exact solution is unavailable, the errors in Figure 1 represent differences from the $N = 1024$ numerical reference solution computed with the same time step, rather than total errors with respect to the exact solution.

4.2. Fractional dynamics. We next investigate the effect of the fractional order α on the solution dynamics. The computational domain is $\Omega = (-1, 1) \times (-1, 1)$, and we use $N_x = N_y = 128$. The final time is $T = 0.05$, and the time step is $\Delta t = T/200$. We consider three fractional orders, $\alpha = 0.001, 0.5, 0.999$. The initial condition is given by

$$u(x, y, 0) = \sin\left(\frac{2\pi(x - L_x)}{R_x - L_x}\right) \sin\left(\frac{2\pi(y - L_y)}{R_y - L_y}\right).$$

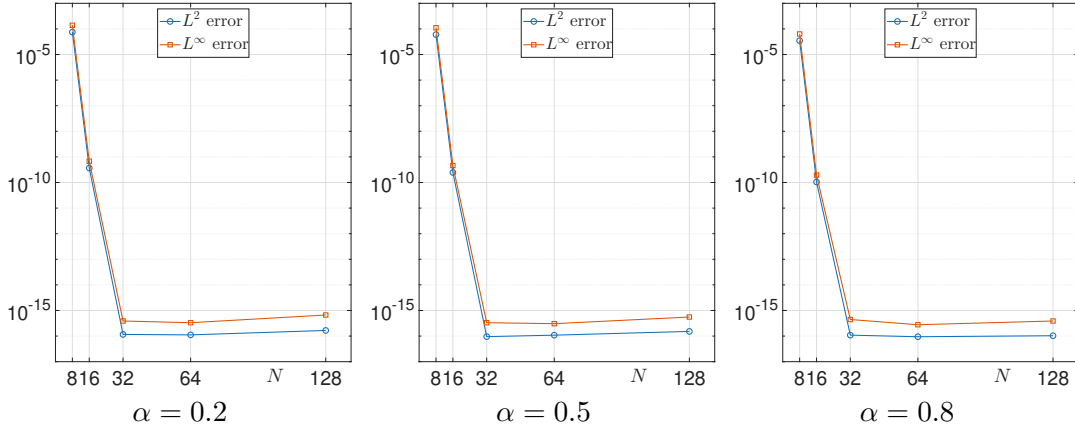


Figure 1: Spatial convergence errors at $T = 0.05$ for $\alpha = 0.2, 0.5$, and 0.8 . The time step is fixed at $\Delta t = 10^{-4}$ for all computations. The numerical solution with $N = 1024$ at the same time step is used as the reference solution.

This initial condition satisfies the homogeneous Dirichlet boundary condition.

To compare the decay of the solutions, we define the maximum absolute value at each time level by

$$M(t_n) = \max_{i,j} |u_{ij}^n|.$$

Figure 2(b) shows the time evolution of $M(t)$ for the three values of α . In all cases, the maximum absolute value decreases with time. However, the decay speed depends on the fractional order. A smaller value of α gives a slower decrease, while the solution decays faster as α approaches one.

The second row of Figure 2 shows the numerical solutions at $t = 0.05$. The overall spatial shape is similar for the three cases, but their amplitudes are different. In particular, the solution decays more slowly for smaller α . The solution with $\alpha = 0.001$ has the largest amplitude, while the solution with $\alpha = 0.999$ has the smallest one. This behavior is also consistent with the curves of $M(t)$ in Figure 2(b).

5. CONCLUSION

In this study, we developed a DST-based numerical method for the two-dimensional normalized time-fractional diffusion equation with homogeneous Dirichlet boundary conditions. The normalized time-fractional derivative was discretized on a uniform time grid, and the spatial derivatives were treated by the sine spectral method. The Laplace operator becomes diagonal after the DST. As a result, each spectral mode can be updated independently. This makes the numerical procedure simple and avoids the direct solution of a large two-dimensional linear system.

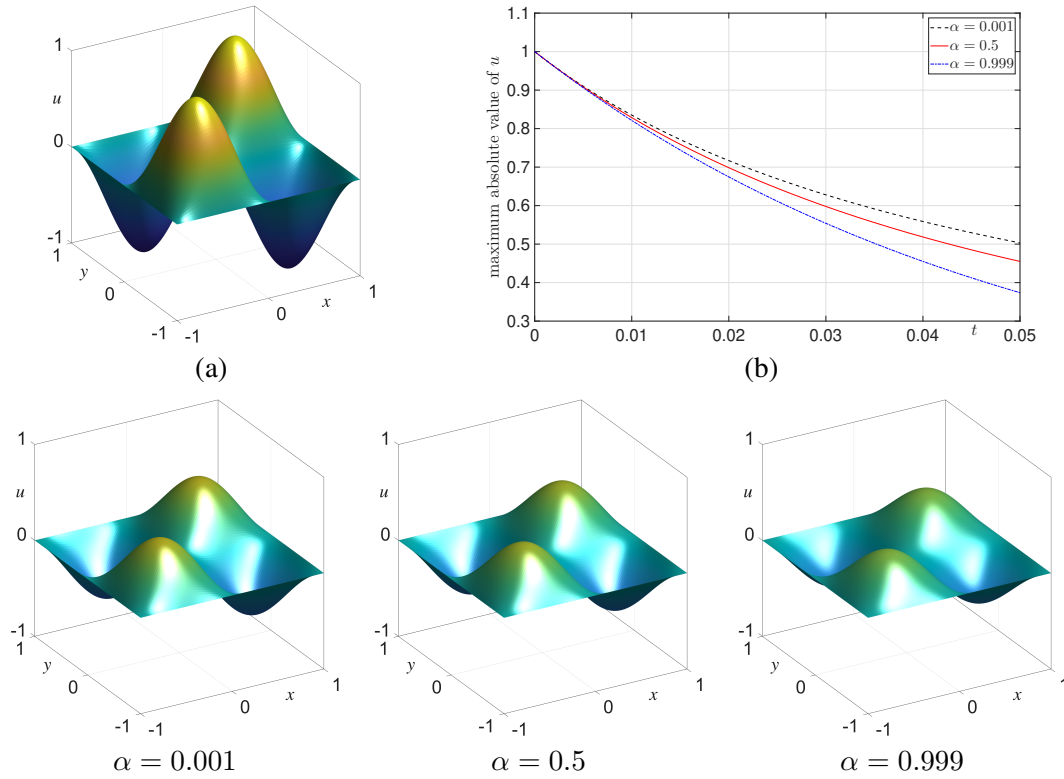


Figure 2: (a) The initial condition. (b) Temporal evolution of the maximum absolute value of u . The second row shows the numerical solutions at $t = 0.05$ for $\alpha = 0.001, 0.5, 0.999$.

We also studied the stability and convergence of the proposed method. The stability analysis showed that the method is unconditionally stable in the discrete ℓ^2 norm. For sufficiently smooth solutions, the temporal convergence order is $2 - \alpha$, while the spatial approximation has spectral accuracy. The numerical experiments agreed well with the theoretical results. For $\alpha = 0.2, 0.5$, and 0.8 , the observed temporal convergence rates were close to 1.8, 1.5, and 1.2, respectively. The spatial errors decreased rapidly and reached the level of machine precision as the number of sine modes increased. The numerical results also showed that the fractional order changes the decay speed of the solution. A smaller value of α produced slower decay in the present simulations.

The proposed approach can be extended in several directions. For example, DST-based methods can be considered for other normalized time-fractional equations with homogeneous Dirichlet boundary conditions. Extensions to higher-dimensional problems and nonlinear models are also possible.

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